On the additive theory of prime numbers II

Patrick Cegielski, Denis Richard, Maxim Vsemirnov

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September 20, 2006

Abstract

The undecidability of the additive theory of primes (with identity) as well as the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$, where p_n denotes the $(n+1)$-th prime, are open questions. As a possible approach, we extend the latter theory by adding some extra function. In this direction we show the undecidability of the existential part of the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$, where r_n is the remainder of p_n divided by n in the euclidian division.

Résumé

L'indécidabilité de la théorie additive des nombres premiers ainsi que de la théorie $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$, où p_n désigne le $(n+1)$-ième premier, sont deux questions ouvertes. Nous étendons cette dernière théorie en lui ajoutant une fonction supplémentaire et nous montrons l'indécidabilité de la théorie $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$, où r_n désigne le reste de p_n de la division euclidienne de p_n par n, et même de sa seule partie existentielle.

Introduction - The additive theory of primes contains longtime open classical conjectures of Number Theory, as famous Goldbach's binary conjecture or twin primes conjecture, and so on. Some authors provided [BJW,BM,LM] conditional proofs (through Schinzel's Hypothesis [SS]) of the undecidability of the additive theory of primes $\mathrm{Th}(\mathbb{N}, +, \mathbb{P})$, where $\mathbb{P}$ is the set of all primes. Weakening the problem by strengthenning this theory, we introduced [CRV] the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$, where p_n is the $(n+1)$-th prime, and posed the problem of its (un)decidability. As usual for a language containing a function symbol, we suppose it contains identity. Note that $\mathbb{P}$ is existentially definable within $\langle \mathbb{N}, n \mapsto p_n \rangle$, hence $\mathrm{Th}(\mathbb{N}, +, \mathbb{P})$ is a subtheory of $\mathrm{Th}(\mathbb{N}, +, n \mapsto p_n)$. At the moment, the undecidability of the latter theory is still an open question, and our approach in [CRV] was to consider several approximations of the function $n \mapsto p_n$ as, for instance, $n \lfloor \log n \rfloor$ and on this way we showed the undecidability of theories $\mathrm{Th}(\mathbb{N}, +, nf(n))$ for a family of functions f including $\lfloor \log \rfloor$ mentioned above. Another approach consists of extending the language $\{+, n \mapsto p_n\}$ to $\{+, n \mapsto p_n,\ n \mapsto r_n\}$, where r_n is the remainder of p_n divided by n. The main result of this paper is the following:

Theorem 1 *Multiplication is existentially* $\langle \mathbb{N}, +, \mathrm{n} \mapsto \mathrm{p_n},\ \mathrm{n} \mapsto \mathrm{r_n} \rangle$*-definable at first-order.*

This leads to the following (without use of conjectures) result:

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Corollary 1 $\mathrm{Th}_{\exists}(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$ *is undecidable.*

Remark Actually all positive integer constants are existentially $\{+, \mathbb{P}\}$-definable in the following manner:

$$\begin{array}{lcl} x = 0 & \Leftrightarrow & x + x = x; \\ x = 1 & \Leftrightarrow & \exists y(y = x + x \wedge y \in \mathbb{P}); \\ x = 2 & \Leftrightarrow & \exists y(y = 1 \wedge x = y + y); \\ & \vdots & \\ x = n + 1 & \Leftrightarrow & \exists y \exists z(y = n \wedge z = 1 \wedge x = y + z). \end{array}$$

As we mentioned above, $\mathbb{P}$ is existentially definable within the language $\{+, n \mapsto p_n\}$, hence all positive integer constants are also existentially $\{+, n \mapsto p_n\}$-definable. Note, that $n \left\lfloor \frac{p_n}{n} \right\rfloor = p_n - r_n$. We intend to define (section 2, see Lemma 3) $\left\lfloor \frac{p_n}{n} \right\rfloor$ from $+$ and $n \left\lfloor \frac{p_n}{n} \right\rfloor$. Then the strategy will be to define multiplication through the function $n \mapsto cn^2$ (where c is a fixed constant), which is to be proved $\{+, \left\lfloor \frac{p_n}{n} \right\rfloor, n \left\lfloor \frac{p_n}{n} \right\rfloor\}$-definable. Consequently, multiplication will be existentially $\{+, n \mapsto p_n, n \mapsto r_n\}$-definable at first-order.

Remark. In the previous paper [CRV] we consider continuous real strictly increasing functions and their inverses. Since we work with integer parts we have to introduce pseudo-inverses of discrete functions. For such a discrete unbounded function f from $\mathbb{N}$ into $\mathbb{N}$, we define its pseudo-inverse f^{-1} from $\mathbb{N}$ into $\mathbb{N}$ by $f^{-1}(n) = \mu m[f(m+1) > n]$, where μ means "the smallest ... such that". Due to the unboundness of f such an f^{-1} is correctly defined.

1) Some preliminary results in Number Theory

Contrarily to what happens with log, the behavior of $\left\lfloor \frac{p_n}{n} \right\rfloor$ is *a priori* irregular but we shall prove it is not too much chaotic. Namely, we prove:

Proposition 1 *Let us denote the mapping* $n \mapsto \left\lfloor \frac{p_n}{n} \right\rfloor$ *by* f.
1) For $m > n$, *we have* $f(m) - f(n) \geq -1$;
2) For $n \geq 11$, *we have* $f^{-1}(n+1) - f^{-1}(n) > n$.

Proof 1) We use the following estimates for p_n ([RP], p. 249):

$$p_m \geq m \log m + m \log \log m - 1.0072629m \quad \text{for } m \geq 2;$$
$$p_m \leq m \log m + m \log \log m - 0.9385m \quad \text{for } m \geq 7022. \tag{1}$$

For $m > n \geq 7022$, we have $f(m) - f(n) = \left\lfloor \frac{p_m}{m} \right\rfloor - \left\lfloor \frac{p_n}{n} \right\rfloor$
$\geq \frac{p_m}{m} - \frac{p_n}{n} - 1 \geq log(\frac{m}{n}) - log(\frac{logm}{logn}) - 0.9385 + 1.0072629 - 1.$
Hence $f(m) - f(n) \geq -1$ because the sum of the two first terms is positive as is the sum of terms three and four.
If $n < 7022$, one may check the desired inequality by a direct computation.

2) Let m be $f^{-1}(n)$. By the very definition of f^{-1}, the equality $m = f^{-1}(n)$ is equivalent to the conjunction of the two following conditions:

$$\begin{cases} \left\lfloor \dfrac{p_{m+1}}{m+1} \right\rfloor \geq n + 1; \\ \forall k \leq m \quad \left\lfloor \dfrac{p_k}{k} \right\rfloor \leq n. \end{cases} \tag{2}$$

For $k \le 7022$, the maximum of $\frac{p_k}{k}$ is attained for $k = 7012$ and equal to $\frac{p_{7012}}{7012} < 10.102824 < 11$. Consequently, we see that $m = f^{-1}(n) \ge f^{-1}(11) \ge 7022$ and this is the reason why in the hypothesis of Proposition 1, item 2) we assume $n \ge 11$.
To prove the inequality, it is sufficient to prove that for $k = m + n$ we have $\lfloor \frac{p_k}{k} \rfloor \le n + 1$, or in other words,

$$\frac{p_k}{k} < n + 2. \tag{3}$$

Note that for $m \ge 7022$, we have by (2):

$$n + 1 \le \left\lfloor \frac{p_{m+1}}{m+1} \right\rfloor + 1 \le \frac{p_{m+1}}{m+1} + 1 \le \log(m+1) + \log\log(m+1) - 0.07 < m.$$

Consequently it is sufficient – and actually more convenient – to prove a somehow stronger result, namely the same inequality (3) but for $m \ge 7022$ and $m + 1 \le k \le 2m$.
From the second estimate of (1) we have, since $k \ge m \ge 7022$, the following inequalities:

$$\begin{aligned} \tfrac{p_k}{k} &< \log k + \log\log k - 0.9385 \\ &\le \log 2m + \log\log 2m - 0.9385 \\ &= \log m + \log\log m + \log 2 + \log(1 + \tfrac{\log 2}{\log m}) - 0.9385; \end{aligned}$$

using the first estimate of (1) and $\frac{\log 2}{\log m} \le \frac{\log 2}{\log 7022}$, we have:

$$\log m + \log\log m - 1.0072629 \le \frac{p_m}{m};$$

consequently:

$$\frac{p_k}{k} \le \frac{p_m}{m} + 0.07 + \log 2 + \log(1 + \frac{\log 2}{\log 7022}) \le \frac{p_m}{m} + 1$$

by an easy computation and finally, due to (2), we obtain $\frac{p_k}{k} < n + 2$. □

Item 1) of previous proposition emphasizes on the fact that $f : n \mapsto \lfloor \frac{p_n}{n} \rfloor$ is "almost" increasing and item 2) shows that the difference $f^{-1}(n+1) - f^{-1}(n)$ is big enough with respect to n. This suggests to introduce a new class of functions, containing f, for which we prove that the existential part of the theory $\mathrm{Th}(\mathbb{N}, +, n \mapsto nf(n))$ is undecidable.

2) The class $C(k, d, n_0)$ and some its properties

2.1) The class $C(k, d, n_0)$

Let $k \ge 0$ be a fixed nonnegative integer. We shall say f is k-*almost increasing* if and only if

$$\forall y \ge x[f(y) - f(x) \ge -k]. \tag{4}$$

In this sense 0-almost increasing means increasing (not necessarily strictly) and $n \mapsto \lfloor \frac{p_n}{n} \rfloor$ is 1-almost increasing (due to Proposition 1).

Still looking at $n \mapsto \lfloor \frac{p_n}{n} \rfloor$, we intend to consider functions whose pseudo-inverse is defined and asymptotically increases quickly enough with respect to its argument. Let us say that f^{-1} has at least $(1/d)$-linear difference, if

$$\exists n_0 \in \mathbb{N} \forall n \ge n_0 [f^{-1}(n+1) - f^{-1}(n) > \frac{n}{d}]. \tag{5}$$

In fact, for $\lfloor \frac{p_n}{n} \rfloor$, the constant d is 1 and $n_0 = 11$, but results and proofs hold for an arbitrary (fixed) d.

Now let us definite the class $C(k,d,n_0)$ as the set of functions from $\mathbb{N}$ into $\mathbb{N}$ satisfying conditions (4) of being *k-almost increasing* and (5) of having its pseudo-inverse with an *at least $(1/d)$-linear difference.*

In order to prove FUNDAMENTAL LEMMA of section 3, whose Theorem 1 is a corollary, we show some properties of the class $C(k,d,n_0)$. Firstly, in section 2.2 we present in three lemmas these properties and comment them. Afterwards, in section 2.3, we prove them.

2.2) Properties of $C(k,d,n_0)$

Lemma 1 *For any function $f \in C(k,d,n_0)$ the following items hold:*

(i) For any $n \geq n_0$, we have $f^{-1}(n+d) - f^{-1}(n) > n$;

(ii) For any $n \geq n_0+1$, the set $\{x \in \mathbb{N} \mid f(x) = n\}$ is nonempty;

(iii) For any $n \geq n_0+1$, the equality $f(x) = n$ implies

$$x > \frac{1}{2d}[(n-1)(n-2) - n_0(n_0-1)].$$

Lemma 2 *If $f \in C(k,d,n_0)$ and $f(x) = n \geq n_0$, then for any c such that $1 \leq c \leq n$, we have:*

$$-k \leq f(x+c) - f(x) \leq k+d. \tag{6}$$

Lemma 3 *For any $f \in C(k,d,n_0)$, let $x_0 = f^{-1}(2+4d+n_0^2+k)$.*
Consider $\tilde{f} : [x_0+1, +\infty[\cap\mathbb{N} \longrightarrow \mathbb{N}$ with $\tilde{f}(x) = f(x)$. Then $\tilde{f}$ is existentially definable at first-order within $\langle \mathbb{N}, +, 1, x \mapsto xf(x) \rangle$.

<u>Remarks</u> 1) Item (i) of Lemma 1 provides a linear lower bound of values of f^{-1} when difference of arguments is the parameter d of the considered class.
Item (ii) of the same lemma insure that f is asymptotically onto, and item (iii) gives a quadradic lower bound for solutions of the equation $f(x) = n$, that we need in section 3.

2) Actually, as the reader will see within the proof, Lemma 1 does not use condition (4) of being k-almost increasing.

3) Lemma 2 provides asymptotical bounds for the difference of two values of f with arguments taken in a short interval with respect to the values of these arguments. Refering to the previous Lemma 1 we see that n is at most $\sqrt{2dx+n_0^2}+2$.

4) Lemma 3 generalizes the situation of the main result of the previous paper [CRV] of the same authors when $\lfloor \log n \rfloor$ was "extracted", *i.e.* defined, from $+$ and $n\,\lfloor \log n \rfloor$.

2.3) Proofs of the three Lemmas

Proof of Lemma 1 (i) By condition (5):

$$\begin{aligned} f^{-1}(n+d) - f^{-1}(n) &= [f^{-1}(n+d) - f^{-1}(n+d-1)] \\ &\quad +[f^{-1}(n+d-1) - f^{-1}(n+d-2)] \\ &\quad +\ldots \\ &\quad +[f^{-1}(n+1) - f^{-1}(n)] \\ &> \tfrac{n+d-1}{d} + \tfrac{n+d-2}{d} + \ldots + \tfrac{n}{d} \\ &> n. \end{aligned}$$

(ii) If there was no x such that $f(x) = n$, we would have $f^{-1}(n) = f^{-1}(n-1)$. But $f^{-1}(n) > f^{-1}(n-1)$ according to condition (5).

(iii) By definition of $f^{-1)}$, we have: $x > f^{-1}(n-1)$.
As in (i), we have:

$$\begin{array}{rcl} f^{-1}(n-1) - f^{-1}(n_0) & = & [f^{-1}(n-1) - f^{-1}(n-2)] \\ & & + \ldots \\ & & +[f^{-1}(n_0+1) - f^{-1}(n_0)] \\ & > & \frac{n-2}{d} + \frac{n_0}{d} + \ldots + \frac{n}{d} \\ & = & \frac{(n-2)(n-1) - n_0(n_0+1)}{2d}. \end{array}$$

and the result. □

Proof of Lemma 2 The left-hand side of the inequality is an immediate consequence of the very definition of a k-almost increasing function. For proving the right-hand side, note that, using k-almost increasing property of f together with $f(x) = n$, we obtain:

$$\max_{y \leq x} f(y) \leq f(x) + k = n + k,$$

so that $f^{-1}(n+k) \geq x$, by the definition of f^{-1}. By previous Lemma 1, item (i) and the latter inequality, we have:

$$f^{-1}(n+k+d) > f^{-1}(n+k) + n + k \geq x + n + k \geq x + n \geq x + c$$

since $1 \leq c \leq n$. Using again the definition of f^{-1}, we see that $f(x+c) \leq n + k + d = f(x) + k + d$ and we are done. □

Proof of Lemma 3 To define $\tilde{f}$ within the structure $\langle \mathbb{N}, +, x \mapsto xf(x) \rangle$ we shall make use of the inequality:

$$0 \leq f(x) < x$$

together with the remainder of $f(x)$ modulo $x+1$, which we must define in the considered structure.

Fact 1.- $f(x) < x$.

By the definition of f^{-1}, we have $f(x_0+1) > k+2+4d+n_0^2$ and by the k-almost increasing property we deduce, for $x \geq x_0 + 1$,

$$n = f(x) \geq f(x_0+1) - k > 2 + 4d + n_0^2. \tag{7}$$

Hence $\frac{n-2}{2d} > 2$.
From (7), we obtain $n > n_0 + 1$ so that by Lemma 1, item (iii), we have:

$$x > \frac{1}{2d}[(n-1)(n-2) - n_0(n_0-1)],$$

hence:

$$x > 2(n-1) - \frac{n_0(n_0-1)}{2d} > 2(n-1) - n_0^2 = n + (n - 2 - n_0^2) > n = f(x). \ \Box\Box$$

Fact2.- *We have*:

$$f(x) \equiv (x+1)f(x+1) - xf(x) \pmod{x+1}. \tag{8}$$

It is sufficient to note that $(x+1)f(x+1) - xf(x) = f(x) + (x+1)[f(x+1) - f(x)]$. □□

We are still unable to define general congruences, fortunately here the difference $|f(x+1) - f(x)|$ is bounded, namely,

$$|f(x+1) - f(x)| \leq k + d, \tag{9}$$

due to Lemme 2, with $c = 1$. This suggests to introduce the notion of a restricted congruence, namely, for a, b, m in $\mathbb{N}$ and some fixed integer c, we define $a \equiv_c b (\operatorname{mod} m)$ by:

$$\bigvee_{h=0}^{c} \{[a = b + \underbrace{m + \cdots + m}_{\text{h times}}] \vee [b = a + \underbrace{m + \cdots + m}_{\text{h times}}]\}.$$

Obviously, the first-order latter formula is expressible within the structure $\langle \mathbb{N}, + \rangle$, since c is fixed. The congruence (8) and inequality (9) provide together the following restricted congruence:

$$f(x) \equiv_{k+d} (x+1)f(x+1) - xf(x) (\operatorname{mod} x+1),$$

which is a definition of $f(x)$ within $\langle \mathbb{N}, +, 1, x \mapsto xf(x) \rangle$ since $1 \leq f(x) < x$. Finally, we provide explicitely an existential first-order definition of f, namely:

$$[x > x_0 \wedge y = f(x)] \leftrightarrow$$

$$\{x > x_0 \wedge y \leq x \wedge \bigvee_{h=0}^{k+d} [(y + xf(x) = (x+1)f(x+1) + \underbrace{(x+1) + \cdots + (x+1)}_{\text{h times}})$$

$$\vee ((x+1)f(x+1) = y + xf(x) + \underbrace{(x+1) + \cdots + (x+1)}_{\text{h times}})]\}.$$

3) Fundamental Lemma and the proof of the Main Theorem

In order to prove the undecidability of $\mathrm{Th}(\mathbb{N}, n \mapsto p_n, n \mapsto r_n)$, we prove a more general result, namely:

Lemma 4 (Fundamental Lemma) *For any* $f \in C(k, d, n_0)$ [see §2]*, multiplication is existentially* $\{+,\ 1,\ x \mapsto xf(x)\}$*-definable at first-order.*

As shown by Y. MATIYASEVICH, the existential true theory of numbers is exactly the set of arithmetical relations, which are definable by diophantine equations. Therefore the negative solution of the 10-th Hilbert's problem [MY] implies the following corollary.

Corollary 2 *The existential theory* $\mathrm{Th}_\exists(\mathbb{N},\ +,\ 1,\ x \mapsto xf(x))$ *is undecidable.*

Proof of Lemma 4 It suffices to show that for some constants c and n_1 the function $n \mapsto cn^2$ from $[n_1, +\infty[\cap \mathbb{N}$ into $\mathbb{N}$ is $\{+,\ 1,\ x \mapsto xf(x)\}$-definable. More precisely, we shall take $c = 5d$ and $n_1 = 2 + 5d + n_0^2$. Consider $n \geq n_1$. Since $n_1 > n_0 + 1$, we can apply Lemma 1, item (ii), proving there exists x such that $f(x) = 5dn$. Let x_0 be the same as in Lemma 3, namely $x_0 = f^{-1}(2 + 4d + n_0^2 + k)$. Let us show $x > x_0$. Otherwise $x \leq x_0$, so that by the k-almost increasing property $f(x) \leq f(x_0) - k$, implying, by the definitions of f^{-1} and x_0,

$$f(x) \leq 2 + 4d + n_0^2 + k - k < n_1 < 5dn_1 \leq 5dn = f(x),$$

which is impossible.
Note that $5dn$ is $\{+\}$-definable as the sum of $5d$ terms equal to n (d is a fixed constant).
Now thanks to Lemma 3, an x such that $f(x) = 5dn$ is $\{+, 1, x \mapsto xf(x)\}$-definable.
On the other hand:

$$(x+n)f(x+n)-xf(x) = (x+n)[f(x+n)-f(x)]+nf(x) = (x+n)[f(x+n)-f(x)]+5dn^2.$$

By Lemma 2 applied to $c = n$, we have $|f(x+n) - f(x)| \leq k + d$, so that:

$$5dn^2 \equiv_{k+d} (x+n)f(x+n) - xf(x) (\bmod x+n). \tag{10}$$

According to Lemma 1 and item (iii) since $f(x) = 5dn$ and $5dn > n_1 > n_0 + 1$ the inequalities $n \geq n_1 > n_0^2$ and:

$$\begin{aligned} x+n &> \frac{(5dn-1)(5dn-2)}{2d} - \frac{n_0(n_0-1)}{2d} + n \\ &> \frac{25d^2n^2 - 15nd}{2d} > 5dn^2 \end{aligned} \tag{11}$$

hold.
Using (10) and (11), a similar argument as in Lemma 3 shows that the function $n \mapsto 5dn^2 = cn^2$ with domain $[n_1, +\infty[\cap\mathbb{N}$ is existentially $\{+, 1, x \mapsto xf(x)\}$-definable. By a routine argument, multiplication is clearly existentially $\{+, 1, x \mapsto xf(x)\}$-definable. $\square$

Proof of the Main-Theorem We remind the reader that 1 was existentially $\{+, \mathbb{P}\}$ and $\{+, n \mapsto p_n\}$-defined in the introduction.
We also noted that $n\left\lfloor \frac{p_n}{n} \right\rfloor = p_n - r_n$ and $n \mapsto n\left\lfloor \frac{p_n}{n} \right\rfloor$ belongs to $C(1, 1, 11)$, the latter due to Proposition 1, §1. Then Fundamental Lemma can be applied and multiplication is existentially $\{+, n \mapsto p_n, n \mapsto r_n\}$-definable. $\square$

Conclusion: Our main result is absolute in the sense that does not depend on any conjecture. In order to shed more light on the considered theories $\mathrm{Th}_\exists(\mathbb{N}, +, \mathbb{P})$ and $\mathrm{Th}_\exists(\mathbb{N}, n \mapsto p_n, n \mapsto r_n)$, we recall a conditional result of A. Woods. Let us recall that Dickson's conjecture [DL] claims that if $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ are integers with all $a_i > 0$ and

$$\forall y \neq 1 \exists x [y \not| \prod_{1 \leq i \leq n} (a_i x + b_i)]$$

then there exist infinitely many x such that $a_i x + b_i$ are primes for all i. Let us call *DC* this conjecture, then A. Woods proved [WA]:

If DC is true then the existential theory $\mathrm{Th}_\exists(\mathbb{N}, +, \mathbb{P})$ *is decidable.*

Now, the question is to know whether there is a gap between $\mathrm{Th}_\exists(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$ or whether they are exactly the same. In the case of equality between these two theories, *DC* is false (and hence Schinzel's hypothesis on primes, whose *DC* is the linear case, is also false).

Open problem: *Is* $\mathrm{Th}_\exists(\mathbb{N}, +, \mathbb{P})$ *equal to* $\mathrm{Th}_\exists(\mathbb{N}, +, n \mapsto p_n, n \mapsto r_n)$?

References

[BJW] P.T. Bateman, C.G. Jockusch and A.R. Woods, *Decidability and Undecidability of theories with a predicate for the prime*, **Journal of Symbolic Logic**, vol. 58, 1993, pp.672-687.

[BM] Maurice Boffa, *More on an undecidability result of* Bateman, Jockusch *and* Woods, **Journal of Symbolic Logic**, vol. 63, 1998, p.50.

[CRV] Patrick Cegielski, Denis Richard & Maxim Vsemirnov, *On the additive Theory of Prime Numbers I*, **Proceedings of CSIT'2003** (Computer Science and Information Technologies), September 22-26, 2003, Yerevan, Armenia, 459 p., pp. 80–85.

[DL] L.E. Dickson, *A new extension of* Dirichlet*'s theorem on prime numbers*, **Messenger of Mathematics**, vol. 33 (1903–04), pp. 155–161.

[LM] T. Lavendhomme & A. Maes, *Note on the undecidability of* $\langle\omega, +, P_{m,r}\rangle$, Definability in arithmetics and computability, 61-68. **Cahier du Centre de logique**, Belgium, 11 (2000).

[MY] Yuri Matiyasevich, **Hilbert's tenth Problem**, The MIT Press, Foundations of computing, 1993, XXII+262p.

[RP] Paul Ribenboim, **The new book of Prime records**, Springer, 1996, XIV+541p.

[SS] A. Schinzel & W. Sierpieńsky, *Sur certaines hypothses concernant les nombres premiers*, **Acta Arithmetica**, vol. 4, 1958, 185–208 and 5, 1959, 259.

[WA] Alan Woods, **Some problems in logic and number theory, and their connection**, Ph.D. thesis, University of Manchester, Manchester, 1981.